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Simple Algorithm for Adaptive Refinement of Three-Dimensional Finite Element Tetrahedral Meshes

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A simple strategy to adaptively refine three-dimensional tetrahedral meshes has been implemented. The procedure adaptively refines a crude initial mesh using solution error indicators or other suitable measures. In this paper example problems were remeshed using a refinement ratio determined from an a posteriori error indicator obtained from the finite element solution of the problem. The resulting finite element meshes are found to have a smooth gradient in element size. Aspect ratios are calculated to determine the quality of each element, and a smoothing procedure is employed to improve the element aspect ratio. Example meshes are included to show the adaptive nature of the remesher when applied over several solution cycles.

Introduction

IN the current work a simple strategy to adaptively refine three-dimensional tetrahedral meshes is described. The three-dimensional strategy is similar to the two-dimensional refinement discussed in Nambiar et al.[1] The extension to three dimensions, however, adds considerable difficulty.[2] Although several initial mesh generation methods are discussed in the literature,[3–7] few three-dimensional refinement techniques have been published. A three-dimensional refinement scheme for tetrahedral elements has been presented by Rivara and Levin[8] in which an algorithm which subdivides elements by bisecting the longest edge is discussed. However, in the process, nonconforming elements are formed, and a second procedure to identify nonconforming elements and proceed with further subdivision is presented.[8]

In the current method, element division is also based on the longest edge, but nonconforming elements are not generated, and unlike the process described in Ref. 8 where a local refinement is performed, the method of the present paper is applied globally and, thus, produces a smooth transition in element size over the entire mesh. The list of elements to be refined is first sorted in ascending order based on the longest edge length of each element. Thus, the last element of the sorted list is guaranteed to be the element with the longest edge of those requiring subdivision. On subdivision, the list is updated, and the refinement ratios are proportionally distributed to each child element. This procedure guarantees that nonconforming elements will not be generated, and it has been observed that the poorest aspect ratio of elements generated by this subdivision is very close to the worst aspect ratio of the initial mesh. Results discussed by Rivara and Levin[8] also confirm this observation.

Quantitative measures of the quality of tetrahedral elements are discussed by Cavendish et al.,[3] Buratynski,[9] and Johnston and Sullivan.[10] For comparison purposes, the values presented by the authors in Refs. 3, 9, and 10 were normalized for the present work by assuming that the best tetrahedron, an equilateral, has a quality number of 1.0. Cavendish et al.[3] reported meshes with up to 10% of the elements with aspect ratios below 0.01 (0.03 on the normalized scale used in our comparisons). Buratynski[9] reported average aspect ratios of 0.224 and 0.225 for two examples with a lowest aspect ratio of 0.03 (0.09 on the normalized scale). Johnston and Sullivan[10] reported that less than 0.2–1.2% of the elements in three different meshes have aspect ratios less than 0.01 (0.03 on the normalized scale). Johnston and Sullivan[10] also state that elements with aspect ratio less than 0.01 (0.03 on the normalized scale) are considered degenerate.

Refinement Procedure

Refinement Density and Refinement Ratio

The mesh refinement procedure presented in this paper is based on the error estimator and refinement ratio presented in Ref. 11. If $\|\hat{u}\|$ is the energy norm calculated for the entire finite element model, and $\|e\|$ is the error in energy norm, the corrected energy norm $\|u\|$ is

$$\|u\|^2 = \|\hat{u}\|^2 + \|e\|^2$$

If $\bar{\eta}$ is the maximum permissible relative error requested for the finite element solution, a mean permissible error for an element can be calculated from

$$\bar{e}_m = \bar{\eta}\sqrt{\|u\|^2/m}$$

where m is the number of elements.

The refinement ratio ξ_i for each element is then defined by

$$\xi_i = \|e_i\|/\bar{e}_m$$

where e_i is the solution error estimate associated with the ith element. An element with a refinement ratio greater than 1.0 must be subdivided in proportion to the magnitude of ξ_i and the analysis repeated. That is, an element with $\xi_i = 3$ must be divided into three elements. Although it is ideal to achieve a refinement ratio of 1.0 for all of the elements, it is impossible to obtain a value of exactly 1.0 for all of elements in a practical problem unless elements are resized by adjusting nodal locations.[12]

Each refinement ratio ξ_i is converted into a refinement density which enables the refinement ratio to be distributed in terms of the volume of each element on a material-by-material basis. The refinement density of an element is calculated as

$$\text{refinement density} = \frac{\text{refinement ratio}}{\text{element volume}}$$

The element refinement densities are then assigned to each node of the mesh by averaging the refinement densities of the elements which share a node.

$$\text{nodal refinement density} = \frac{\Sigma\ \text{(surrounding element refinement densities)}}{\text{number of surrounding elements}}$$

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The nodal distribution may be used to interpolate for refinement densities of nodes that are created in the subdivision process discussed next.

Subdivision Procedure

The elements with refinement ratios greater than 1.0 are sorted based on the longest edge length of the elements in the error list. The last element in the sorted list is the element with the longest edge length and is the first candidate for subdivision. To avoid the generation of nonconforming elements, it is essential to divide all of the elements that surround the longest edge of the element which has been selected for subdivision. This situation is shown in Fig. 1.

We determine if all surrounding elements are already included in the list of elements to be subdivided. If an adjacent element does not appear in the list, this element must be inserted into the list at a position based on its longest edge length. This situation is shown in Fig. 2. This process is carried out recursively until it is ensured that each element sharing the longest edge of the element targeted for subdivision is included in the list. The subdivision process itself is then begun. Figure 3 shows a flowchart of the mesh refinement procedure.

The subdivision starts with the identification of the edge to be divided. If the edge is in the interior of the object, a new node is inserted at its midpoint. If the edge happens to be a boundary edge, the inserted node is moved to a location based on the geometry of the boundary surface on which the edge lies. The new node is given the average of the nodal refinement densities of the two parent nodes which defined the edge before bisection. Next, two new elements are created, and their refinement densities are calculated as the average of the nodal refinement densities of the four nodes of the element. The nodal refinement densities are converted to refinement ratios, and any child element whose refinement ratio is greater than 1.0 is inserted into the sorted refinement list based on its longest edge length. The process of subdivision is continued until the refinement ratio list contains all of the elements with ratios less than or equal to 1.0.

Postsubdivision Procedure—Local Smoothing

The mesh may then be smoothed by considering the polyhedron that encloses a newly generated internal node. A Laplacian smoothing, the movement of the newly created node to the centroid of the surrounding polyhedron, is carried out if the new node is in the interior. The element aspect ratio (quality number) is calculated as the ratio of the radius of the circumscribed sphere to inscribed sphere, and these values are normalized so that an equilateral tetrahedron

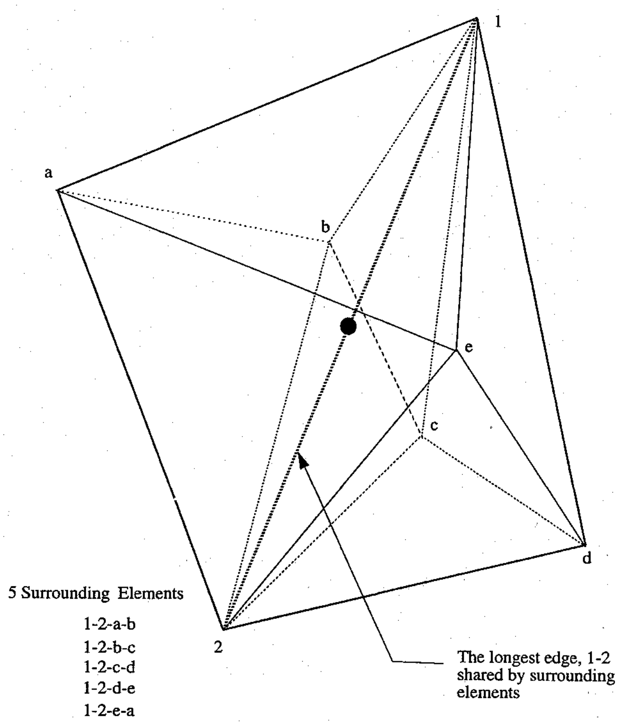

Fig. 1 Edge division, case 1.

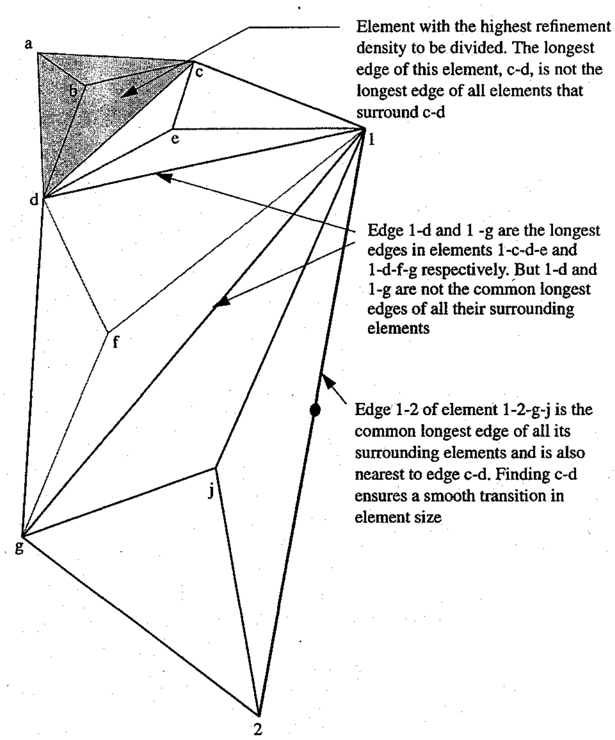

Fig. 2 Edge division, case 2.

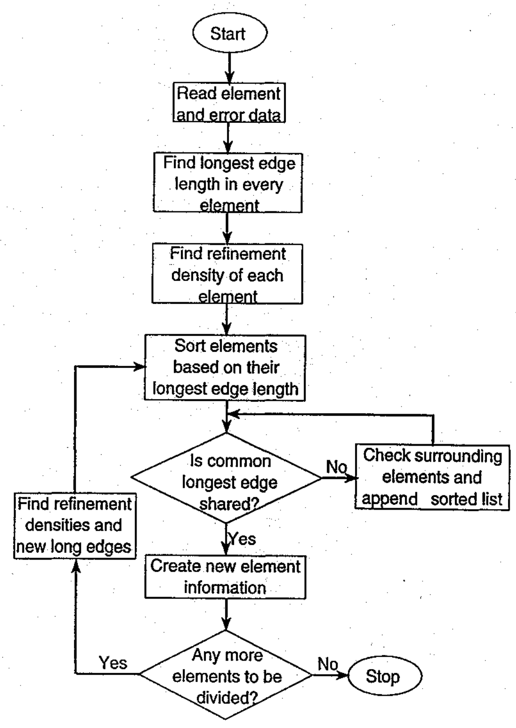

Fig. 3 Flowchart for the adaptive refinement procedure.

will have aspect ratio of 1.0, and a sliver-like tetrahedron will have an aspect ratio approaching 0.0. The aspect ratio is calculated as

$$\text{aspect ratio} = \text{quality number}$$

$$= K\frac{\text{radius of circumscribed sphere}}{\text{radius of inscribed sphere}}$$

where, $K = \frac{1}{3}$ is the normalizing constant defined by the equilateral tetrahedron.

Data Structure for Remeshing

The remeshing procedure is efficiently implemented by maintaining an appropriate data structure. Structures are maintained for nodes, elements, surfaces, boundary edges, and surface loads. The structures used for storing the finite element model are discussed in the Appendix. When a node is created, its coordinates and refinement density are generated. The boundary conditions of newly created nodes are obtained from the boundary conditions of the parent nodes. The element structure is updated for each new element created. The longest edge length and refinement density information are also updated in the element structure. The boundary edge structure carries the data describing the surface type and the nodes that make up the boundary edge. The surface structure is essential to maintain information about the bounding faces and surfaces of the model.

Example Problems

Two example problems illustrate the ideas just discussed. The results for these problems were obtained using a finite element method (FEM) solver, an a posteriori error estimator, and the tetrahedral element refinement procedure. Linear strain elements described in Refs. 13 and 14 were used in the solver procedure. The solution process is described as follows.

1) Develop a simple initial mesh.

2) Obtain the FEM solution and error estimate.

3) If all $\xi_i < 1$ and the global error estimate goal has been reached, quit.

4) Else, refine the mesh and repeat the solution.

Figure 4 shows the first problem, a thin plate with a hole under axial pressure, which is solved iteratively to reach an estimated global error level of less than 5%. The initial mesh and the final mesh using linear strain tetrahedral (LST) elements are shown in Fig. 5. The histogram plots of element quality obtained from adaptive mesh refinement are shown in Fig. 6. The distribution of the aspect ratios in the initial mesh for this problem is also shown in Fig. 6. A comparison of the distribution of element aspect ratios in the thin plate with a hole after adaptive and uniform mesh refinement is shown in Fig. 7.

The second problem, a thick walled cylinder under internal pressure is defined in Fig. 8. The problem was solved and remeshed

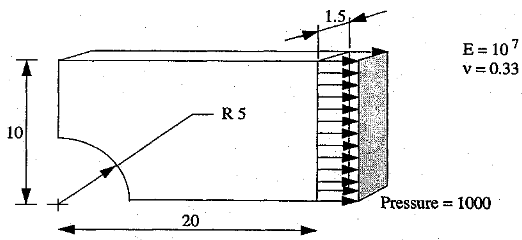

Fig. 4 Thin plate with a circular hole under axial pressure.

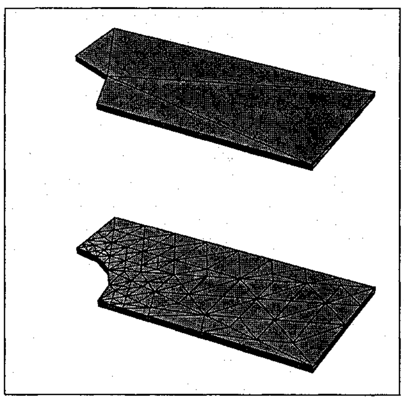

Fig. 5 Initial and adaptive mesh for the plate with a hole problem.

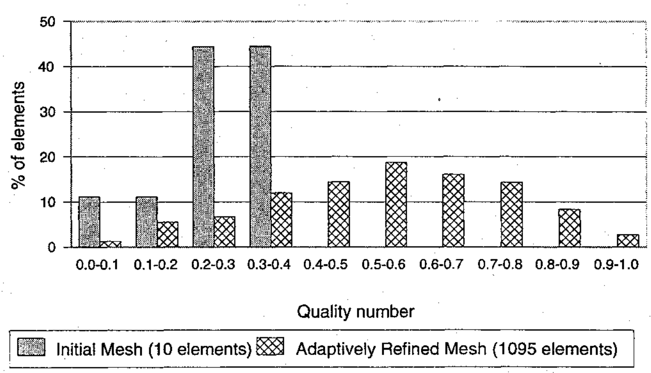

Fig. 6 Element quality numbers for initial and adaptive mesh of plate with a hole.

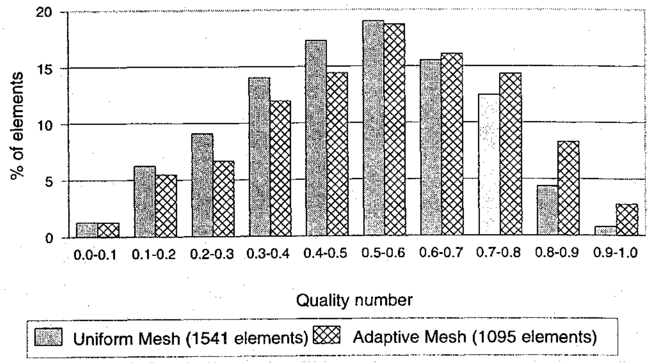

Fig. 7 Element quality numbers for adaptive and uniform mesh of plate with a hole.

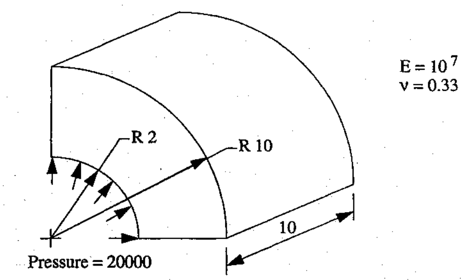

Fig. 8 Problem of thick walled cylinder subjected to internal pressure.

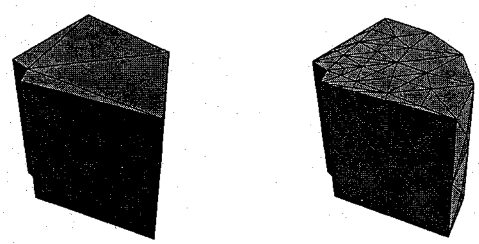

Fig. 9 Initial and adaptive mesh for thick cylinder.

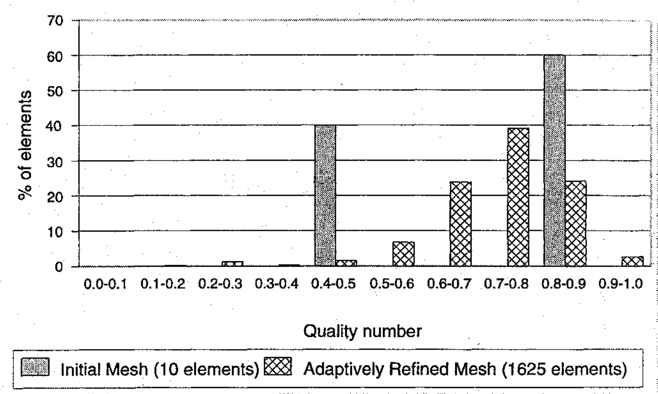

Fig. 10 Element quality numbers for initial and adaptive mesh of thick cylinder.

Table 1 Comparison of quality number, strain energy, and estimated global error: thin plate with a hole

Item		Mesh, LST elements: Initial	Final
Number of elements in mesh		10	1095
Number of refinements		12	
Degrees of freedom		96	5157
Bandwidth		17	1857
Quality number	Best	0.361	0.977
	Worst	0.083	0.033
Strain energy		5.886	7.012
Estimated global error,[a] %		12.13	1.95

[a]Requested error was 5%.

Table 2 Comparison of quality number, strain energy, and estimated global error: thick wall cylinder

Item		Mesh, LST elements: Initial	Final
Number of elements in mesh		10	1625
Number of refinements		9	
Degrees of freedom		105	7539
Banwidth		17	2687
Quality number	Best	0.879	0.978
	Worst	0.484	0.179
Strain energy		1203.6	1735.5
Estimated global error,[a] %		14.2	4.1

[a]Requested error was 5%.

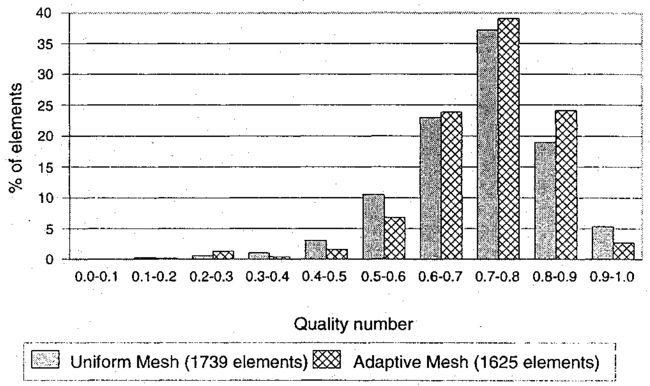

Fig. 11 Element quality numbers for adaptive and uniform mesh of thick cylinder.

several cycles to reach an estimated global error level below 5%. The initial mesh and the final mesh for the thick walled cylinder are shown in Fig. 9. The distribution of the element aspect ratios within the thick walled cylinder model subsequent to adaptive and uniform mesh refinement are shown in Figs. 10 and 11, respectively. A summary of the results of analysis by adaptive refinement is presented in Tables 1 and 2.

Results and Conclusion

A method for automatically refining a tetrahedral element mesh has been described and examples of its use are presented. An important feature of this process is that no nonconforming elements are produced, and thus when applied, a globally smooth mesh results.

An examination of the computed results shows that the distribution of aspect ratios of the elements in the thick walled cylinder are skewed toward +1.0 by adaptive refinement. Because the ratios of length to thickness and width to thickness are inherently poor for the thin plate with a hole problem, the distribution of aspect ratios is central. However, the aspect ratio distribution for the final mesh shifts toward +1.0 with refinement as shown in Figs. 6 and 7. In the final plate with a hole model, the percentage of elements which have aspect ratios less than 0.1 is about 1.28%, and about 7% of the elements have aspect ratio less than 0.2.

Figures 7 and 11 show that the uniform and adaptive meshing produce comparable element quality distribution. However, by uniform refinement a much greater number of elements are required to reach an estimated global error level of less than 5% as compared to the adaptive refinement procedure. This is because the element distribution in the uniformly refined model does not follow any stress or strain gradient as it does with an adaptive mesh.

In a two-dimensional triangular mesh it is possible to determine in advance the worst aspect ratio that will result from the subdivision of an initial mesh by bisection of the longest edge of the element.[15] No such proof is known to exist when this procedure is extended to a three-dimensional tetrahedral mesh. But a comparison of the aspect ratio of elements obtained by the tetrahedral mesh refinement procedure discussed in this paper with previously published results[3,9,10] indicates the effectiveness of the present procedure in producing good quality elements.

We have found that Laplacian smoothing marginally improves the aspect ratio of the tetrahedra in the model. This has also been reported by Lo.[5] For problems solved thus far using the current procedure, no degenerate elements were formed.

A major advantage of the adaptive refinement procedure discussed here is that a simple initial mesh is sufficient, and nonconforming elements are not generated during refinement. In addition, our results indicate that the aspect ratio of the element does not significantly deteriorate when using this algorithm for mesh refinement. Thus this procedure, when used with linear strain or higher order elements, provides an effective approach to the efficient solution of three-dimensional models. Although the results presented in this paper were developed using an h-type refinement process, the method described can be used in conjunction with hp-type and s-type[12] mesh refinements as well.

Appendix: Data Structures

```
typedef struct mesh3d_tet_nodes {
    integer bc_x, bc_y, bc_z;
    integer parent_node_1, parent_node_2;
    double x, y, z;
    double temperature;
    double nod_ref_dens, mult_fac;
} TYP_TET_NODES;

typedef srtuct mesh3d_tet_elements {
    integer long_edge_id;
    integer nodes[4];
    integer material;
    integer type;
    integer quality;
    double volume, err_ratio;
    double long_edge_len;
    double circumsphere_vol, insphere_vol;
} TYP_TET_ELEMENTS;

typedef struct mesh3d_tet_force {
    integer node;
    double mag_x, mag_y, mag_z;
} TYP_TET_FORCE;

typedef struct mesh3d_tet_pressure {
    integer element;
    integer face_node[3];
    double mag_press;
} TYP_TET_PRESSURE;

typedef struct mesh3d_tet_shear {
    integer element;
    integer face_node[3];
    double mag_shear;
    double shear_vec_x, shear_vec_y, shear_vec_z;
} TYP_TET_SHEAR;

typedef struct mesh3d_tet_enforced_displacement {
    integer node;
    double displ_x, displ_y, displ_z;
} TYP_TET_DISPLACEMENT;
```

```
typedef struct mesh3d_tet_surface_param {
    integer type;
    double x[2], y[2], z[2];
    double radius[2];
} TYP_TET_SURFACE;

typedef struct mesh3d_tet_boundary_edge {
    integer node[2];
    integer surface[2];
    integer wing_edge[4];
} TYP_TET_BOUND_EDGE;
```

Acknowledgments

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